Relevant S is Undecidable

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Plan for the talk

- Relevant $\mathbf{S}$: What is it and why is it(s decision problem) interesting?
- Proof technique: undecidability through tiling
- A simpler, yet illustrative proof: no FMP

Defining Relevant S

Definition (language and semantics)

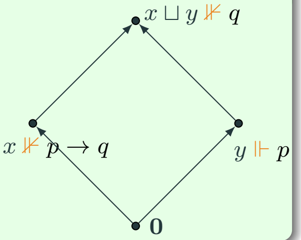
The language is given by

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi.$$

and the semantics of '$\to$' is:

$$x \Vdash \varphi \to \psi \quad \text{iff} \quad \forall y\colon y \Vdash \varphi \Rightarrow x \sqcup y \Vdash \psi$$

Example



Definition (frames and validity)

A frame $\mathfrak{F} = (S, \sqcup, \mathbf{0})$ is a **semilattice** $(S, \sqcup)$ with least element $\mathbf{0} \in S$; i.e.,

- *Commutative*: $x \sqcup y = y \sqcup x$,
- *Associative*: $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$,
- *Idempotent*: $x \sqcup x = x$.
- *Identity (least element)*: $x \sqcup \mathbf{0} = x$.

Equivalently, it is a **partial order** with all **binary joins** and a **least element**.
Finally, a formula φ is valid iff $\mathfrak{M}, \mathbf{0} \Vdash \varphi$ for all models $\mathfrak{M}$.

Problem of concern: *Is* $\mathbb{S}$*'s validity problem decidable?*

But first: why is this interesting?

Motivation

Setting

- $\mathbf{S}$ was introduced by Urquhart (1972, 1973).
- It's a close relative of $\mathbf{R}$ and its positive reduct $\mathbf{R}^+ = \mathbf{R}_{\{\wedge,\vee,\to\}}$.
  - In fact, $\mathbf{S}_{\{\wedge,\to\}} = \mathbf{R}_{\{\wedge,\to\}}$.
- Relevant logics are substructural logics, thus sharing close affinities with, e.g., linear logic.
  - For instance, $\mathbf{R}^+$ is positive linear logic + distribution of additive connectives + contraction.
  - As a rule of thumb: linear logics + contraction = relevant logics.

Why is $\mathbf{S}$'s decision problem interesting?

- Omitting disjunction, the logic $\mathbf{S}_{\{\wedge,\to\}}$ is decidable.
- $\mathbf{S}$ is closely connected to positive relevant $\mathbf{R}^+$, which is undecidable.
  - This, among more, was shown by Urquhart (1984), but $\mathbf{S}$ eluded these techniques.
  - Eventually, this led Urquhart (2016) to conjecture that $\mathbf{S}$ is decidable.

Overarching theme: Understanding the decidability/undecidability boundary in the realm of substructural logics.

Proof method: tiling

- A (Wang) tile is a square with colors on each side.
- The tiling problem: given any finite set of tiles $\mathcal{W}$, determine whether each point in the quadrant $\mathbb{N}^2$ can be assigned a tile from $\mathcal{W}$ such that neighboring tiles share matching colors on connecting sides.
- The tiling problem was introduced by Wang (1963) and proven undecidable by Berger (1966).

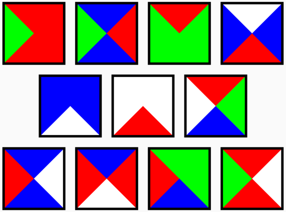


Figure 1: Wang tiles

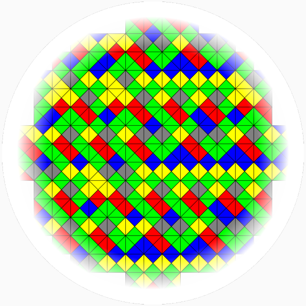


Figure 2: A tiling of the plane

Figures taken from: https://en.wikipedia.org/wiki/Wang_tile

Main theorem

Theorem

$\mathbf{S}$ is undecidable.

Proof idea.

For each finite set of tiles $\mathcal{W}$, we construct a formula $\psi_{\mathcal{W}}$ such that $\mathcal{W}$ tiles the quadrant if and only if $\psi_{\mathcal{W}}$ is refutable. $\square$

Guide to Paper, and Summary

Guide to paper: The conference paper also contains a proof that $\mathbf{S}$ lacks the FMP. If interested, I recommend reading this first, as it is considerably simpler than the undecidability proof, yet effectively illustrates some of the same key ideas.

Summary and further work

- $\mathbf{S}$ is undecidable.
  - Proven via tiling
  - Themes of undecidability proof echoed in the simpler no-FMP proof
- Similar ideas recently applied to solve open problems in the area of modal and temporal logics.[1]
- Future work includes decision problems in the vicinity of linear logic, separation logic, and relevant logic.
  - For instance, is 'contraction-free' $\mathbf{S}$ decidable?

[1] including the longstanding open problem of the decidability of hyperboolean modal logic, as posed by Goranko and Vakarelov (1999).

References I

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Goranko, V. and D. Vakarelov (1999). "Hyperboolean Algebras and Hyperboolean Modal Logic". In: *Journal of Applied Non-Classical Logics* 9.2-3, pp. 345–368. DOI: 10.1080/11663081.1999.10510971 (cit. on p. 10).

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Urquhart, A. (1984). "The undecidability of entailment and relevant implication". In: *Journal of Symbolic Logic* 49, pp. 1059–1073 (cit. on p. 6).

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Wang, H. (1963). "Dominoes and the ∀∃∀ case of the decision problem". In: *Mathematical Theory of Automata*, pp. 23–55 (cit. on p. 8).

Weiss, Y. (2021). "A Conservative Negation Extension of Positive Semilattice Logic Without the Finite Model Property". In: *Studia Logica* 109, pp. 125–136 (cit. on p. 15).

Thank you!

Proof method: tiling

Theorem

Given any logic whose language contains $\{\wedge, \vee, \rightarrow\}$, if its $\{\wedge, \vee, \rightarrow\}$-reduct extends $\mathbf{S}$ and is valid on $(\mathcal{P}(\mathbb{N}), \cup, \varnothing)$, then it is undecidable. In particular, $\mathbf{S}$ is undecidable.

Proof idea.

For each finite set of tiles $\mathcal{W}$, we construct a formula $\psi_{\mathcal{W}}$ such that $\mathcal{W}$ tiles the quadrant if and only if $\psi_{\mathcal{W}}$ is refutable. $\square$

Lemma

If $\psi_{\mathcal{W}}$ is refutable (in a semilattice), then $\mathcal{W}$ tiles $\mathbb{N}^2$.

Lemma

If $\mathcal{W}$ tiles $\mathbb{N}^2$, then $\psi_{\mathcal{W}}$ is refutable (in $(\mathcal{P}(\mathbb{N}), \cup, \varnothing)$).

Relevant S is undecidable: Proof idea

Theorem: S is undecidable.

We cover the no-FMP proof instead, since it is considerably simpler than the undecidability proof, yet effectively illustrates some of the same key ideas.[2]

Theorem: S lacks the FMP.

Proof. We show that the formula ψ_∞ from the paper only is refuted by infinite models.

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2 Additionally, it addresses an open problem (as recently raised in Weiss 2021)

Refuting model